

# CATEGORIFIED INVARIANTS AND THE BRAID GROUP

JOHN A. BALDWIN AND J. ELISENDA GRIGSBY

ABSTRACT. We investigate two “categorified” braid conjugacy class invariants, one coming from Khovanov homology and the other from Heegaard Floer homology. We prove that each yields a solution to the word problem but not the conjugacy problem in the braid group.

## 1. INTRODUCTION

Recall that the  $n$ -strand braid group  $B_n$  admits the presentation

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| \geq 2, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i - j| = 1 \end{array} \right\rangle,$$

where  $\sigma_i$  corresponds to a positive half twist between the  $i$ th and  $(i + 1)$ st strands. Given a word  $w$  in the generators  $\sigma_1, \dots, \sigma_{n-1}$  and their inverses, we will denote by  $\sigma(w)$  the corresponding braid in  $B_n$ . Also, we will write  $\sigma \sim \sigma'$  if  $\sigma$  and  $\sigma'$  are conjugate elements of  $B_n$ . As with any group described in terms of generators and relations, it is natural to look for combinatorial solutions to the *word* and *conjugacy problems* for the braid group:

- (1) Word problem: Given words  $w, w'$  as above, is  $\sigma(w) = \sigma(w')$ ?
- (2) Conjugacy problem: Given words  $w, w'$  as above, is  $\sigma(w) \sim \sigma(w')$ ?

The fastest known algorithms for solving Problems (1) and (2) exploit the Garside structure(s) of the braid group (cf. [6] for a survey and [10] for an implementation). In addition, any faithful representation of  $B_n$  for which the images of the generators and the product rule can be described combinatorially – for example, the Lawrence-Krammer representation [19, 17, 5] – provides a solution to Problem (1).

The present work is an attempt to understand what two popular combinatorial link homology theories coming from representation theory and symplectic geometry – namely, Khovanov homology and link Floer homology – can tell us about Problems (1) and (2). Both theories are powerful enough to detect the unknot [33, 18]. Can they detect the trivial braid? Can they distinguish braid conjugacy classes?

In this short note, we explain how to extract braid conjugacy class invariants from both theories and we prove that each of these invariants provides a solution to Problem (1) but not (naively) to Problem (2). The approaches to Problems (1) and (2) described here are, at present, more computationally involved than the solutions alluded to at the top. In that sense, our results are primarily of theoretical interest. Perhaps more tractable solutions to these problems can be obtained along similar lines in the future as faster algorithms are discovered for computing Khovanov and link Floer homology.

---

JAB was partially supported by NSF grant number DMS-1104688.

JEG was partially supported by NSF grant number DMS-0905848 and NSF CAREER award DMS-1151671.

Below, we provide a brief description of the “categorified” braid conjugacy class invariants studied in this note. Both are invariants of isotopy classes of oriented links in the solid torus complement of a neighborhood of an oriented unknot  $B \subset S^3$ . We will think of  $B$  as the compactification of the oriented  $z$ -axis in  $\mathbb{R}^3$ ,

$$B = \{(r, \theta, z) \mid r = 0\} \cup \{\infty\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3,$$

and the complement  $S^3 - N(B)$  as the product

$$A \times I = \{(r, \theta, z) \mid r \in [1, 2], \theta \in [0, 2\pi), z \in [0, 1]\}$$

of an annulus  $A$  with the interval  $I = [0, 1]$ .

Given a braid  $\sigma \in B_n$  (read and oriented, by convention, from top to bottom), we will imagine its oriented closure  $\widehat{\sigma}$  as living in  $A \times I$  such that  $\widehat{\sigma}$  intersects every disk

$$D_t = \{(r, \theta, z) \mid \theta = t\} \cup \{\infty\}$$

positively in  $n$  points, where  $D_t$  is oriented so that  $B = \partial D_t$ . Note that  $\widehat{\sigma} \subset A \times I$  is well-defined up to isotopy and, moreover, that conjugate braids give rise to isotopic closures. Thus, any invariant of isotopy classes of links in  $A \times I$  gives rise to a braid conjugacy class invariant.

The central objects of study in this paper are the sutured annular Khovanov homology  $SKh(\widehat{\sigma} \subset A \times I)$  (cf. Section 2.1) and the link Floer homology  $\widehat{HFL}(\widehat{\sigma} \cup B)$  (cf. Section 2.2). Both are invariants of the conjugacy class of  $\sigma$  per the observation above. As mentioned earlier, we will prove that both can be used to distinguish unequal braids (cf. Theorem 1) but that neither always distinguishes non-conjugate braid pairs (cf. Theorem 2 and Corollary 1). These invariants share other structural features, including a relationship with the Burau representation

$$\Psi : B_n \rightarrow GL_n(\mathbb{Z}[T^{\pm 1}])$$

(cf. Remarks 2.1 and 2.3). In particular, the graded Euler characteristic of  $\widehat{HFL}(\widehat{\sigma} \cup B)$  is (more or less) the characteristic polynomial of  $\Psi(\sigma)$ . We will show, however, that the homology contains more braid information than does the polynomial in general (cf. Proposition 2.1).

**Acknowledgements.** The authors wish to thank Christian Blanchet, who asked the question that initiated this investigation, Stephan Wehrli, for a number of interesting conversations, and Matt Hedden and Liam Watson, for sharing their paper [13], which provided a key argument in our proof that  $SKh(\widehat{\sigma})$  detects the trivial braid.

## 2. CATEGORIFIED BRAID CONJUGACY CLASS INVARIANTS

In this section, we briefly recall the construction of sutured annular Khovanov homology and some basic features of link Floer homology. We will assume the reader is familiar with ordinary Khovanov homology and knot Floer homology. All chain complexes and homology theories considered in this paper are with coefficients in  $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ .

**2.1. Sutured Annular Khovanov Homology.** Sutured annular Khovanov homology was originally defined in [1] as a categorification of the Kauffman bracket skein module of  $A \times I$ . It was studied further in [38, 11], where a connection with sutured Floer homology was

discovered (hence, the name). The theory associates to an oriented link  $\mathbb{L} \subset A \times I$  a triply-graded vector space

$$SKh(\mathbb{L}) = \bigoplus_{i,j,k} SKh^i(\mathbb{L}; j, k),$$

which is an invariant of the oriented isotopy class of  $\mathbb{L} \subset A \times I$ .

Its construction starts with a projection of  $\mathbb{L}$  onto the annulus  $A \times \{1/2\}$ . This projection may be viewed as a planar diagram  $D$  in  $S^2 - \{X, O\}$ , where  $X$  and  $O$  are basepoints in  $S^2$  corresponding to the inner and outer boundary circles of  $A \times \{1/2\}$ . Forgetting the data of the  $X$  basepoint temporarily, we may think of  $D$  as a planar diagram in  $\mathbb{R}^2 = S^2 - \{O\}$  and form the ordinary bigraded Khovanov complex

$$CKh(D) = \bigoplus_{i,j} CKh^i(D; j)$$

from a cube of resolutions of  $D$  in the usual way. Here,  $i$  and  $j$  are the homological and quantum gradings, respectively.<sup>1</sup> The basepoint  $X$  gives rise to a filtration on  $CKh(D)$ , and  $SKh(\mathbb{L})$  is defined to be the (co)homology of the associated graded object.

To define this filtration, we choose an oriented arc from  $X$  to  $O$  missing all crossings of the diagram  $D$ . As described in [12, Sec. 4.2], the generators of  $CKh(D)$  are in one-to-one correspondence with *enhanced* (i.e., *oriented*) resolutions. We define the “ $k$ ” grading of an oriented resolution to be the algebraic intersection number of this resolution with our oriented arc, up to some overall shift. Roberts proves ([38, Lem. 1]) that the Khovanov differential does not increase this extra grading. One therefore obtains a filtration,

$$0 \subseteq \dots \subseteq \mathcal{F}_{n-1}(D) \subseteq \mathcal{F}_n(D) \subseteq \mathcal{F}_{n+1}(D) \subseteq \dots \subseteq CKh(D),$$

where  $\mathcal{F}_n(D)$  is the subcomplex of  $CKh(D)$  generated by oriented resolutions with  $k$  grading at most  $n$ . Let

$$\mathcal{F}_n(D; j) = \mathcal{F}_n(D) \cap \bigoplus_i CKh^i(D; j).$$

The sutured annular Khovanov homology groups of  $\mathbb{L}$  are defined to be

$$SKh^i(\mathbb{L}; j, k) := H^i \left( \frac{\mathcal{F}_k(D; j)}{\mathcal{F}_{k-1}(D; j)} \right).$$

The lemma below follows directly.

**Lemma 2.1.** *There is a spectral sequence whose  $E_1$  term is  $SKh(\mathbb{L} \subset A \times I)$  and whose  $E_\infty$  term (ignoring the  $k$  grading) is isomorphic to  $Kh(\mathbb{L} \subset S^3)$ . Moreover, the  $d_n$  differential shifts the  $(i, j, k)$  multi-grading by  $(1, 0, -n)$ .*

Given a braid  $\sigma$ , we will be interested in  $SKh(\widehat{\sigma})$ , where  $\widehat{\sigma} \subset A \times I$  is the oriented closure of  $\sigma$  as described in the Introduction.

**Remark 2.1.** It is shown in [2] that the summand of  $SKh(\widehat{\sigma})$  in the next-to-top  $k$  grading is equal to the Hochschild homology of the braid bimodule constructed by Khovanov and Seidel in [16]. This bimodule detects the trivial braid [16, Cor. 1.2] (and its action categorifies the Burau representation [16, Prop. 2.8]). These facts together with Theorem 2 imply that a categorical group representation may be faithful even while its Hochschild homology is not faithful on conjugacy classes. Similar results have been obtained in the Heegaard Floer setting by Hedden and Watson [13] in combination with Lipshitz, Ozsváth and Thurston [20, Thm. 14], [21].

<sup>1</sup>The grading,  $i$ , is really a *cohomological* grading, as the Khovanov differential *increases* it by 1.

On the other hand, Theorem 1 suggests the following question.

**Question 2.1.** Suppose one has a faithful weak action, in the sense of [16, Def. 2.6], of a group  $G$  on a (derived) category of modules over an  $(A_\infty)$  algebra  $A$ , where the functor associated to an element  $g \in G$  is given by taking a (derived) tensor product with a (derived equivalence class of) bimodule  $\mathcal{M}_g$ . Let  $HH(A, \mathcal{M}_g)$  denote the Hochschild homology of  $A$  with coefficients in  $\mathcal{M}_g$ . Does

$$HH(A, \mathcal{M}_g) = HH(A, \mathcal{M}_1)$$

necessarily imply that  $g = 1$ ?

**2.2. Link Floer Homology.** Link Floer homology was defined in [35] as a generalization of knot Floer homology and a categorification of the multi-variable Alexander polynomial. One version of the theory associates to an oriented link  $\mathbb{L} \subset S^3$  expressed as a union of  $k$  sublinks,  $\mathbb{L} = L_1 \cup \dots \cup L_k$ , a graded vector space

$$\widehat{HFL}(\mathbb{L}) = \bigoplus_{d, A_{L_1}, \dots, A_{L_k}} \widehat{HFL}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_k})$$

which is an invariant of the oriented isotopy class of  $\mathbb{L} \subset S^3$ . Here,  $d$  is the Maslov grading and the  $A_{L_i}$  are the Alexander gradings associated to the sublinks  $L_i$ .<sup>2</sup> When  $k = 1$ ,  $\widehat{HFL}(\mathbb{L})$  as described here is isomorphic to the knot Floer homology  $\widehat{HFK}(\mathbb{L})$  after an overall shift of the Maslov grading [31, Thm. 1.1].

Link Floer homology enjoys many symmetries. To begin with,  $\widehat{HFL}(\mathbb{L})$  is supported in Alexander multi-gradings that are symmetric about the origin in  $\mathbb{R}^k$ ,

$$(1) \quad \widehat{HFL}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_k}) \cong \widehat{HFL}_{d-2\sum_i A_{L_i}}(\mathbb{L}; -A_{L_1}, \dots, -A_{L_k}).$$

Moreover, if  $\mathbb{L}'$  is the link obtained by replacing  $L_i$  with  $-L_i$ , then

$$(2) \quad \widehat{HFL}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_i}, \dots, A_{L_k}) \cong \widehat{HFL}_{d-2A_{L_i}+l_i}(\mathbb{L}'; A_{L_1}, \dots, -A_{L_i}, \dots, A_{L_k}),$$

where  $l_i = \text{lk}(L_i, \mathbb{L} - L_i)$ . Similarly, if  $m(\mathbb{L})$  denote the mirror of  $\mathbb{L}$ , then

$$(3) \quad \widehat{HFL}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_k}) \cong \widehat{HFL}_{-d+|\mathbb{L}|-1}(m(\mathbb{L}); A_{L_1}, \dots, A_{L_k}).$$

See [24, Sec. 5] and [31, Sec. 8] for discussions of these symmetries.

The link Floer homology of  $\mathbb{L}$  is the homology of a chain complex defined in terms of a multi-pointed Heegaard diagram for  $\mathbb{L}$ . In an abuse of notation, we will denote this complex by  $\widehat{CFL}(\mathbb{L})$ . For each sublink  $L_i$ ,  $\widehat{CFL}(\mathbb{L})$  can be realized as the associated graded object of a filtration on some complex  $C(\mathbb{L} - L_i)$ , where

$$H_*(C(\mathbb{L} - L_i)) \cong \widehat{HFL}(\mathbb{L} - L_i) \otimes V^{\otimes |L_i|}$$

up to a shift of the Alexander multi-grading, for  $k \geq 2$ . Here,  $V$  is the triply-graded vector space  $\mathbb{F} \oplus \mathbb{F}$  whose summands are supported in Maslov gradings 0 and  $-1$  and Alexander multi-grading  $(0, \dots, 0) \in \mathbb{R}^{k-1}$ . This leads to the following lemma (the claim about grading shifts follows immediately from the discussions in Subsections 3.7 and 8.1 of [31]).

**Lemma 2.2.** *For  $k \geq 2$ , there is a spectral sequence whose  $E_1$  term is  $\widehat{HFL}(\mathbb{L})$  and whose  $E_\infty$  term (ignoring the  $A_{L_i}$  grading) is isomorphic to the vector space obtained from  $\widehat{HFL}(\mathbb{L} - L_i) \otimes V^{\otimes |L_i|}$  by shifting each Alexander grading  $A_{L_j}$  by  $\frac{1}{2}\text{lk}(L_j, L_i)$ .*

<sup>2</sup>Typically, one associates an Alexander grading to each *component* of  $\mathbb{L}$ . The theory we describe here is obtained by “flattening” the multi-grading associated to each sublink  $L_i$  into a single grading by summing.

The following describes a similar spectral sequence in the case that  $k = 1$  (cf. [31]).

**Lemma 2.3.** *For  $k = 1$ , there is a spectral sequence whose  $E_1$  term is  $\widehat{HFL}(\mathbb{L})$  and whose  $E_\infty$  term is a rank  $2^{|\mathbb{L}|-1}$  vector space.*

We now restrict our attention to the case  $\mathbb{L} = \widehat{\sigma} \cup B$ . Note that  $\widehat{HFL}(\widehat{\sigma} \cup B)$  is an invariant of the oriented isotopy class of  $\widehat{\sigma} \subset A \times I$ .

**Remark 2.2.** That  $\widehat{HFL}(\widehat{\sigma} \cup B)$  is the homology of the associated graded object of a filtration on a complex which computes  $\widehat{HFL}(\widehat{\sigma}) \otimes V$  (up to an Alexander grading shift) closely parallels the relationship between  $SKh(\widehat{\sigma} \subset A \times I)$  and  $Kh(\widehat{\sigma} \subset S^3)$  described in the previous subsection.

**Remark 2.3.** The fact that link Floer homology categorifies the multi-variable Alexander polynomial, combined with an older result of Morton [27], implies that  $\widehat{HFL}(\widehat{\sigma} \cup B)$  categorifies the characteristic polynomial  $\det(\lambda - \Psi(\sigma))$ , where

$$\Psi : B_n \rightarrow GL_n(\mathbb{Z}[T^{\pm 1}])$$

is the Burau representation of  $B_n$ . More precisely,

$$\sum_{d, A_{\widehat{\sigma}}, A_B} (-1)^d \cdot \widehat{HFL}_d(\widehat{\sigma} \cup B; A_{\widehat{\sigma}}, A_B) \cdot T^{A_{\widehat{\sigma}}} \cdot \lambda^{A_B} = \det(\lambda - \Psi(\sigma)) \cdot (T - 1)^{|\widehat{\sigma}|},$$

up to an overall factor of  $\pm T^{m_1} \cdot \lambda^{m_2}$  for some  $m_1, m_2 \in \frac{1}{2}\mathbb{Z}$ .

On the other hand, since the Burau representation is not faithful for  $n \geq 5$  (cf. [26, 23, 4]), Theorem 1.(b) implies that  $\widehat{HFL}(\widehat{\sigma} \cup B)$  contains strictly more information about  $\sigma$  than does  $\det(\lambda - \Psi(\sigma))$ :

**Proposition 2.1.** *For each  $n \geq 5$  there exists a braid  $\sigma \in B_n$  for which*

$$\det(\lambda - \Psi(\sigma)) = \det(\lambda - \Psi(\mathbb{1})) \quad \text{but} \quad \widehat{HFL}(\widehat{\sigma} \cup B) \not\cong \widehat{HFL}(\widehat{\mathbb{1}} \cup B).$$

### 3. TRIVIAL BRAID DETECTION: SOLUTION TO WORD PROBLEM

The goal of this section is to prove that  $SKh(\widehat{\sigma})$  and  $\widehat{HFL}(\widehat{\sigma} \cup B)$  detect the trivial braid and therefore provide solutions to Problem (1), per the theorem below.

**Theorem 1.** *Suppose  $\sigma \in B_n$ .*

- (a) *If  $SKh(\widehat{\sigma}) \cong SKh(\widehat{\mathbb{1}})$ , then  $\sigma = \mathbb{1}$ .*
- (b) *If  $\widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\mathbb{1}} \cup B)$  and  $\sigma$  is a pure braid, then  $\sigma = \mathbb{1}$ .<sup>3</sup>*

*In particular, both  $SKh(\widehat{\sigma})$  and  $\widehat{HFL}(\widehat{\sigma} \cup B)$  can be used to give solutions to the word problem in the braid group.*

*Proof of Theorem 1.(a).* First, some notation: for  $X, Y \subset Z$ , we will denote by  $\Sigma(Z, X)$  the double cover of  $Z$  branched along  $X$  and by  $\widetilde{Y} \subset \Sigma(Z, X)$  the preimage of  $Y$  under the covering map  $\Sigma(Z, X) \rightarrow Z$ .

Each arrow in the diagram below indicates the existence of a spectral sequence with  $E_i$  page (for some  $i < \infty$ ) the source of the arrow and  $E_\infty$  page the target of the arrow. (The sutures used to define  $SFH(-\Sigma(A \times I, \widehat{\sigma}))$  in the upper right are the preimages of two oppositely oriented meridians on the boundary of the solid torus  $A \times I$ ; see [11] for details.)

---

<sup>3</sup>If  $\sigma$  is not a pure braid, which is easy to check by hand, then  $\sigma \neq \mathbb{1}$ .

$$\begin{array}{ccc}
SKh(\widehat{\sigma} \subset A \times I) & \Longrightarrow & \oplus^2 SFH(-\Sigma(A \times I, \widehat{\sigma})) \\
\Downarrow & & \Downarrow \\
Kh(\widehat{\sigma} \subset S^3) & \Longrightarrow & \oplus^2 \widehat{HF}(-\Sigma(S^3, \widehat{\sigma}))
\end{array}$$

Let us first assume that  $n$  is odd. Then  $\widetilde{B}$  is a knot and

$$SFH(-\Sigma(A \times I, \widehat{\sigma})) \cong \widehat{HFK}(-\Sigma(S^3, \widehat{\sigma}), \widetilde{B})$$

(cf. [15], [11]), in which case the vertical arrow on the right is a special case of the spectral sequence from the knot Floer homology of a knot in a 3-manifold to the Heegaard Floer homology of the 3-manifold (cf. [34, 37]). The bottom horizontal arrow represents two copies of the spectral sequence from the reduced Khovanov homology of  $\widehat{\sigma}$  to the Heegaard Floer homology of  $-\Sigma(S^3, \widehat{\sigma})$  constructed by Ozsváth and Szabó in [36]. The top horizontal arrow corresponds to a sutured generalization of this spectral sequence (cf. [38, 11]). Finally, the vertical arrow on the left signifies the spectral sequence described in Lemma 2.1.

Suppose  $SKh(\widehat{\sigma}) \cong SKh(\widehat{1})$ . Then  $SKh(\widehat{\sigma})$  has rank  $2^n$  and is supported in a single  $i$  grading, by direct calculation. This implies that the spectral sequence from  $SKh(\widehat{\sigma})$  to  $Kh(\widehat{\sigma})$  collapses immediately since the higher differentials shift the  $i$  grading on  $SKh(\widehat{\sigma})$  non-trivially. Therefore,  $Kh(\widehat{\sigma})$  also has rank  $2^n$  and is supported in a single  $i$  grading. This, in turn, implies that the spectral sequence from  $Kh(\widehat{\sigma})$  to  $\oplus^2 \widehat{HF}(-\Sigma(S^3, \widehat{\sigma}))$  collapses immediately since the higher differentials in this spectral sequence shift the  $i$  grading non-trivially as well. Summarizing, we have that

$$\text{rk}(SKh(\widehat{\sigma} \subset A \times I)) = \text{rk}(Kh(\widehat{\sigma} \subset S^3)) = \text{rk}(\oplus^2 \widehat{HF}(-\Sigma(S^3, \widehat{\sigma}))).$$

We also know, from the arrows at the top and right in the diagram above, that

$$\text{rk}(Kh(\widehat{\sigma} \subset S^3)) \geq \text{rk}(\oplus^2 \widehat{HFK}(-\Sigma(S^3, \widehat{\sigma}), \widetilde{B})) \geq \text{rk}(\oplus^2 \widehat{HF}(-\Sigma(S^3, \widehat{\sigma}))).$$

Putting these together, it follows that

$$(4) \quad \text{rk}(\widehat{HFK}(-\Sigma(S^3, \widehat{\sigma}), \widetilde{B})) = \text{rk}(\widehat{HF}(-\Sigma(S^3, \widehat{\sigma}))).$$

Note that  $\widetilde{B}$  is a fibered knot in  $\Sigma(S^3, \widehat{\sigma})$  and can be identified with the binding of the open book decomposition of  $\Sigma(S^3, \widehat{\sigma})$  which is lifted from the open book decomposition of  $S^3$  associated to the unknot  $B$ . Each page of the former open book is thus a double cover of the disk branched along  $n$  points; i.e., a surface of genus  $(n-1)/2$ . The monodromy  $\phi(\sigma)$  of  $\widetilde{B}$  depends only on  $\sigma$  and can be easily expressed as a product of Dehn twists around curves in this surface. Moreover, Birman and Hilden prove in [7] that  $\phi(\sigma)$  is isotopic to the identity if and only if  $\sigma = 1$ . The proof of Theorem 1.(a) in the case that  $n$  is odd follows from this fact and (4), together with the following result of Hedden and Watson.

**Proposition 3.1.** [13, Thm. 2] *Suppose  $K \subset Y$  is a fibered knot with monodromy  $\phi$ . If*

$$(5) \quad \text{rk}(\widehat{HFK}(-Y, K)) = \text{rk}(\widehat{HF}(-Y)),$$

*then  $\phi$  is isotopic to the identity.*

We sketch their proof below.

*Proof.* Suppose  $K$  is fibered with monodromy  $\phi$  and fiber  $S$  and that (5) holds. The Heegaard Floer contact invariant  $c(S, \phi) \in \widehat{HF}(-Y)$  can be thought of as the image of a non-zero class in  $\widehat{HFK}(-Y, K)$  under the spectral sequence from  $\widehat{HFK}(-Y, K)$  to  $\widehat{HF}(-Y)$  (cf. [30]). The equality in (5) implies that this spectral sequence collapses immediately and, hence, that  $c(S, \phi)$  is non-zero. The contact structure corresponding to the open book  $(S, \phi)$  is therefore tight, which implies that  $\phi$  is *right-veering* by work of Honda, Kazez and Matić in [14]. On the other hand, note that  $K \subset -Y$  is the fibered knot corresponding to the open book  $(S, \phi^{-1})$ . Since

$$\text{rk}(\widehat{HFK}(Y, K)) = \text{rk}(\widehat{HFK}(-Y, K)) = \text{rk}(\widehat{HF}(-Y)) = \text{rk}(\widehat{HF}(Y)),$$

the same reasoning as before shows that  $c(S, \phi^{-1})$  is non-zero as well. This implies that  $\phi^{-1}$  is right-veering and, hence, that  $\phi$  is *left-veering*. Since  $\phi$  is both right- and left-veering, it is isotopic to the identity.  $\square$

In the case that  $n$  is even, we first embed  $\sigma$  into  $B_{n+1}$  under the natural injection  $B_n \hookrightarrow B_{n+1}$ , and then apply the reasoning above.  $\square$

The proof of Theorem 1.(b) is very similar in spirit to that of Theorem 1.(a).

*Proof of Theorem 1.(b).* For  $\sigma \in B_n$ , Baldwin, Vela-Vick and Vértesi define in [3] a class  $\widehat{t}(\sigma) \in \widehat{HFL}(m(\hat{\sigma}))$  which is an invariant of the transverse link in the tight contact structure on  $S^3$  represented by the braid  $\sigma$  (and agrees, for transverse *knots*, with the invariants defined in [32, 22]). We show below that if  $\sigma$  is a pure braid and  $\widehat{HFL}(\hat{\sigma} \cup B) \cong \widehat{HFL}(\hat{1} \cup B)$ , then  $\widehat{t}(\sigma)$  and  $\widehat{t}(\sigma^{-1})$  are non-zero. Then, by an analogue of Honda, Kazez and Matić's result, this implies that  $\sigma$  is both right- and left-veering, and therefore equal to  $1$ .

Suppose  $\sigma$  is a pure braid and  $\widehat{HFL}(\hat{\sigma} \cup B) \cong \widehat{HFL}(\hat{1} \cup B)$ . We first compute the latter. Let us denote  $1$  by  $1_n$  to indicate that it is the trivial braid on  $n$ -strands. Note that  $\hat{1}_n \cup B$  is isotopic to link gotten by taking the connected sum of the positive Hopf link  $\hat{1}_1 \cup B$  with itself  $n$  times along the component  $B$ . As computed in [31, Sec. 12] (cf. also [25, Sec. 4]),

$$\widehat{HFL}(\hat{1}_1 \cup B) \cong (V_1 \otimes V_2)[1/2, 1/2].$$

Here,  $V_i$  is the triply-graded vector space  $\mathbb{F} \oplus \mathbb{F}$  whose first and second summands are supported in Maslov gradings 0 and  $-1$  and Alexander bi-gradings  $(0, 0)$  and  $-e_i$ , where  $e_i$  is the  $i$ th standard basis vector in  $\mathbb{R}^2$ . The  $[1/2, 1/2]$  indicates that we have shifted the Alexander bi-grading by  $(1/2, 1/2)$ . Our initial assumption and the Künneth formula in [31] imply that

$$(6) \quad \widehat{HFL}(\hat{\sigma} \cup B) \cong \widehat{HFL}(\hat{1}_n \cup B) \cong \widehat{HFL}(\hat{1}_1 \cup B)^{\otimes n} \cong (V_1^{\otimes n} \otimes V_2^{\otimes n})[n/2, n/2].$$

To compute  $\widehat{t}(\sigma)$ , we need to know the link Floer homology of  $m(\hat{\sigma} \cup -B)$ . From (6) together with the formulae in (2) and (3), it follows that

$$\widehat{HFL}(m(\hat{\sigma} \cup -B)) \cong (V_1^{\otimes n} \otimes V_2^{\otimes n})[n/2, n/2]$$

as well. Lemma 2.2 then implies that there is a spectral sequence with  $E_1$  term

$$\widehat{HFL}(m(\hat{\sigma} \cup -B))[-n/2, 0] \cong (V_1^{\otimes n} \otimes V_2^{\otimes n})[0, n/2]$$

and whose  $E_\infty$  term (ignoring the  $A_{m(-B)}$  grading) is isomorphic to  $\widehat{HFL}(m(\hat{\sigma})) \otimes V$ , where  $V$  is the bi-graded vector space  $\mathbb{F} \oplus \mathbb{F}$  whose summands are supported in Maslov gradings 0



and  $-1$  and  $A_{m(\hat{\sigma})}$  grading  $0$ . To reduce clutter, we introduce the notation

$$\begin{aligned} H(\hat{\sigma}, B) &:= \widehat{HFL}(m(\hat{\sigma} \cup -B))[-n/2, 0], \\ H(\hat{\sigma}) &:= \widehat{HFL}(m(\hat{\sigma})) \otimes V. \end{aligned}$$

Note that the top Maslov graded piece  $H_{top}$  (in Maslov grading  $-n$ ) of the portion of  $H(\hat{\sigma}, B)$  in the bottommost  $A_{m(-B)}$  grading  $-n/2$  has rank one. The element  $\hat{t}(\hat{\sigma})$  admits a characterization in terms of the generator of  $H_{top}$  (cf. [3, Section 6]). In particular,  $\hat{t}(\hat{\sigma})$  is non-zero if and only if  $H_{top}$  survives in the spectral sequence from  $H(\hat{\sigma}, B)$  to  $H(\hat{\sigma})$ .

Note that  $H(\hat{\sigma}, B)$  is supported in  $A_{m(\hat{\sigma})}$  gradings  $-n \leq A_{m(\hat{\sigma})} \leq 0$ . Since  $H(\hat{\sigma})$  is supported in  $A_{m(\hat{\sigma})}$  gradings that are symmetric about the origin (as discussed in Subsection 2.2) it must be that the portion of  $H(\hat{\sigma}, B)$  in negative  $A_{\hat{\sigma}}$  gradings dies in the spectral sequence. Moreover, the portion in  $A_{\hat{\sigma}}$  grading  $0$  has rank  $2^n$ . Therefore,

$$\text{rk}(H(\hat{\sigma})) \leq 2^n.$$

Now, Lemma 2.3 implies that there is a spectral sequence whose  $E_1$  term is  $H(\hat{\sigma})$  and whose  $E_\infty$  term is a rank  $2^{|m(\hat{\sigma})|}$  vector space. Since  $\sigma$  is a pure braid,  $|m(\hat{\sigma})| = n$ . Thus,

$$\text{rk}(H(\hat{\sigma})) \geq 2^n.$$

It follows that  $\text{rk}(H(\hat{\sigma})) = 2^n$ . It must therefore be the case that the entire portion of  $H(\hat{\sigma}, B)$  in  $A_{m(\hat{\sigma})}$  grading  $0$  survives in the spectral sequence from  $H(\hat{\sigma}, B)$  to  $H(\hat{\sigma})$ . Since  $H_{top}$  is contained in this portion we may deduce that  $\hat{t}(\hat{\sigma})$  is non-zero. Hence,  $\sigma$  is right-veering by [3, Theorem 1.4].

Note that  $\hat{\sigma}^{-1} \cup B$  is isotopic to  $m(-\hat{\sigma} \cup B)$ . The formula in (2) and (3) therefore imply that

$$\widehat{HFL}(\hat{\sigma}^{-1} \cup B) \cong \widehat{HFL}(\hat{\sigma} \cup B) \cong \widehat{HFL}(\hat{\mathbb{1}} \cup B)$$

as well. It follows that  $\hat{t}(\hat{\sigma}^{-1})$  is also non-zero. As before, this implies that  $\sigma^{-1}$  is right-veering and, hence, that  $\sigma$  is left-veering. Since  $\sigma$  is both right- and left-veering, it is equal to  $\hat{\mathbb{1}}$ .  $\square$

**Remark 3.1.** For an alternate proof of Theorem 1.(b), one can show that if  $\sigma$  is pure and  $\widehat{HFL}(\hat{\sigma} \cup B) \cong \widehat{HFL}(\hat{\mathbb{1}} \cup B)$ , then  $\widehat{HFK}(\hat{\sigma}) \cong \widehat{HFK}(\hat{\mathbb{1}})$ . It then follows from [29, Thm. 1.1] that  $\hat{\sigma}$  is the  $n$ -component unlink. But the main result of [8] implies that the only  $n$ -braid whose closure is the  $n$ -component unlink is the trivial braid; hence,  $\sigma = \hat{\mathbb{1}}$ . In particular, to detect the trivial braid, one need only consider only  $\widehat{HFK}(\hat{\sigma})$  (and whether  $\sigma$  is pure). Forthcoming work of Batson and Seed, again combined with [8], implies that  $Kh(\hat{\sigma})$  also detects the trivial braid. On the other hand,  $\widehat{HFL}(\hat{\sigma} \cup B)$  and  $SKh(\hat{\sigma})$  contain more braid conjugacy information than  $\widehat{HFK}(\hat{\sigma})$  and  $Kh(\hat{\sigma})$ , as the latter two depend only on the isotopy class of  $\hat{\sigma}$  as a link in  $S^3$ , and not at all on its embedding in  $S^3 - N(B)$ .

The invariants  $SKh(\hat{\sigma})$  and  $\widehat{HFL}(\hat{\sigma} \cup B)$  can be used to solve the word problem in  $B_n$  as follows. Suppose  $w$  and  $w'$  are words in the generators  $\sigma_1, \dots, \sigma_n$  and their inverses representing the braids  $\sigma(w)$  and  $\sigma(w')$  in  $B_n$ . Let  $\sigma = \sigma(w) \cdot (\sigma(w'))^{-1}$  and note that

$$(7) \quad \sigma = \hat{\mathbb{1}} \iff \sigma(w) = \sigma(w').$$

In particular, if  $\sigma$  is not a pure braid, then  $\sigma(w) \neq \sigma(w')$ . If  $\sigma$  is a pure braid, calculate  $SKh(\hat{\sigma})$  or  $\widehat{HFL}(\hat{\sigma} \cup B)$  and apply Theorem 1 with (7) in mind.



#### 4. TRANSVERSE MIRROR INVARIANCE: NO SOLUTION TO CONJUGACY PROBLEM

In this section, we show that  $SKh(\widehat{\sigma})$  and  $\widehat{HFL}(\widehat{\sigma} \cup B)$  cannot always distinguish non-conjugate braids. This fact is made precise in Corollary 1 of Theorem 2 below. First, some definitions and remarks.

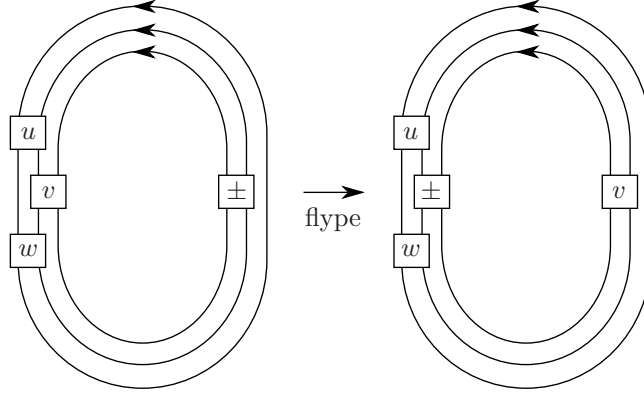


FIGURE 1. Left, a 3-braid closure  $\widehat{\sigma}$ . Each of the boxes labeled  $u, v, w$  represents some number of positive or negative half twists. The box labeled  $\pm$  represents a single positive or negative half twist. Right, the closed braid obtained from  $\widehat{\sigma}$  by a flype. It is oriented isotopic to  $\widehat{\sigma}^r$  in  $A \times I$ .

**Definition 4.1.** Let  $\sigma \in B_n$  and suppose  $\sigma = \sigma(w)$  for some word  $w$  in the generators  $\sigma_1, \dots, \sigma_n$  and their inverses. Then the *reverse* of  $\sigma$  is the braid  $\sigma^r = \sigma(w^r)$ , where  $w^r$  is the word obtained from  $w$  by reversing the order of its letters.

**Remark 4.1.** Thought of as transverse links in the tight contact structure on  $S^3$ , the closure  $\widehat{\sigma}^r$  is the *transverse mirror* of  $\widehat{\sigma}$ , as defined by Ng in [28, Defn. 4.5].

**Theorem 2.** Suppose  $\sigma \in B_n$ . Then

- (a)  $SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}^r)$ ,
- (b)  $\widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\sigma}^r \cup B)$ .

*Proof of Theorem 2.(a).* Let  $D$  and  $D^r$  denote the projections of  $\widehat{\sigma}$  and  $\widehat{\sigma}^r$  onto  $A \times \{1/2\}$  as described in Subsection 2.1. There is an obvious bijection between oriented resolutions of  $D$  and  $D^r$  which preserves the  $(i, j, k)$  triple-grading and commutes with the differentials in the complexes  $CKh(D)$  and  $CKh(D^r)$ . (The projection  $D^r$  is exactly what you see by looking at  $D$  from below and reversing orientations.) It follows that  $SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}^r)$ .  $\square$

*Proof of Theorem 2.(b).* Note that if  $\mathbb{L} = \widehat{\sigma} \cup B$ , then  $\widehat{\sigma}^r \cup B$  is oriented isotopic to  $-\mathbb{L}$  in  $S^3$ . It follows from (1) and (2) that  $\widehat{HFL}(\mathbb{L}) \cong \widehat{HFL}(-\mathbb{L})$  for any oriented link  $\mathbb{L}$ .  $\square$

**Corollary 1.** There exist infinitely many pairs  $(\sigma, \sigma') \in B_3 \times B_3$  such that  $\sigma \not\sim \sigma'$  but

$$SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}') \quad \text{and} \quad \widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\sigma}' \cup B).$$

*Proof.* Suppose  $\sigma$  and  $\sigma'$  are 3-braids for which  $\widehat{\sigma}$  and  $\widehat{\sigma}'$  are related by a flype, as described in [9, Fig. 1.2] and illustrated in Figure 1. Then  $\widehat{\sigma}'$  is clearly oriented isotopic to  $\widehat{\sigma}^r$  in

$A \times I$ . In particular,  $\widehat{\sigma}'$  and  $\widehat{\sigma}$  are transverse mirrors, a fact first observed by Ng in [28]. Theorem 2 then implies that

$$SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}') \quad \text{and} \quad \widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\sigma}' \cup B).$$

On the other hand, Birman and Menasco have enumerated infinitely many conjugacy classes of 3-braids admitting *non-degenerate* flypes; i.e. flypes which do not preserve conjugacy class [9, Tab. 2].  $\square$

## REFERENCES

- [1] Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora. Categorification of the Kauffman bracket skein module of  $I$ -bundles over surfaces. *Algebr. Geom. Topol.*, 4:1177–1210 (electronic), 2004.
- [2] Denis Auroux, J. Elisenda Grigsby, and Stephan M. Wehrli. Sutured Khovanov homology, Hochschild homology, and the Ozsváth-Szabó spectral sequence. *In preparation*, 2012.
- [3] J.A. Baldwin, D.S. Vela-Vick, and V. Vértesi. On the equivalence of Legendrian and transverse invariants in knot Floer homology. 2011.
- [4] Stephen J. Bigelow. The Burau representation is not faithful for  $n = 5$ . *Geom. Topol.*, 3:397–404 (electronic), 1999.
- [5] Stephen J. Bigelow. Braid groups are linear. *J. Amer. Math. Soc.*, 14(2):471–486 (electronic), 2001.
- [6] Joan S. Birman and Tara E. Brendle. Braids: a survey. In *Handbook of knot theory*, pages 19–103. Elsevier B. V., Amsterdam, 2005.
- [7] Joan S. Birman and Hugh M. Hilden. On isotopies of homeomorphisms of Riemann surfaces. *Ann. of Math. (2)*, 97:424–439, 1973.
- [8] Joan S. Birman and William W. Menasco. Studying links via closed braids. V. The unlink. *Trans. Amer. Math. Soc.*, 329(2):585–606, 1992.
- [9] Joan S. Birman and William W. Menasco. A note on closed 3-braids. *Commun. Contemp. Math.*, 10(suppl. 1):1033–1047, 2008.
- [10] Jae Choon Cha and Charles Livingston. Knotinfo table of knot invariants, 2006. Available from <http://www.indiana.edu/~knotinfo/>.
- [11] J. Elisenda Grigsby and Stephan M. Wehrli. Khovanov homology, sutured Floer homology and annular links. *Algebr. Geom. Topol.*, 10(4):2009–2039, 2010.
- [12] J. Elisenda Grigsby and Stephan M. Wehrli. On gradings in Khovanov homology and sutured Floer homology. In *Topology and geometry in dimension three*, volume 560 of *Contemp. Math.*, pages 111–128. Amer. Math. Soc., Providence, RI, 2011.
- [13] Matthew Hedden and Liam Watson. On the geography and botany of knot Floer homology. to appear.
- [14] Ko Honda, William H. Kazez, and Gordana Matić. Right-veering diffeomorphisms of compact surfaces with boundary. *Invent. Math.*, 169(2):427–449, 2007.
- [15] András Juhász. Holomorphic discs and sutured manifolds. *Algebr. Geom. Topol.*, 6:1429–1457 (electronic), 2006.
- [16] Mikhail Khovanov and Paul Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15(1):203–271 (electronic), 2002.
- [17] Daan Krammer. Braid groups are linear. *Ann. of Math. (2)*, 155(1):131–156, 2002.
- [18] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.
- [19] R. J. Lawrence. Homological representations of the Hecke algebra. *Comm. Math. Phys.*, 135(1):141–191, 1990.
- [20] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. Bimodules in bordered Heegaard Floer homology. [math.GT/1003.0598](https://arxiv.org/abs/math.GT/1003.0598), 2010.
- [21] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston. A faithful linear-categorical action of the mapping class group of a surface with boundary. [math.GT/1012.1032](https://arxiv.org/abs/math.GT/1012.1032), 2010.
- [22] P. Lisca, P. Ozsváth, A. Stipsicz, and Z. Szabó. Heegaard Floer invariants of Legendrian knots in contact three-manifolds. 2008.
- [23] D.D. Long and M. Paton. The Burau representation is not faithful for  $n \geq 6$ . *Topology*, 32:439–447, 1993.
- [24] C. Manolescu, P. Ozsváth, Z. Szabó, and D. Thurston. On combinatorial link Floer homology. 11:2339–2412, 2007.

- [25] Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar. A combinatorial description of knot Floer homology. *Annals of Math.*, 169(2):633–660, 2009.
- [26] John Atwell Moody. The Burau representation of the braid group  $b_n$  is unfaithful for large  $n$ . *Bull. Amer. Math. Soc.*, 25:379–384, 1991.
- [27] H. R. Morton. The multivariable Alexander polynomial for a closed braid. In *Low-dimensional topology (Funchal, 1998)*, volume 233 of *Contemp. Math.*, pages 167–172. Amer. Math. Soc., Providence, RI, 1999.
- [28] Lenhard Ng. Combinatorial knot contact homology and transverse knots. *Adv. Math.*, 227(6):2189–2219, 2011.
- [29] Yi Ni. A note on knot Floer homology of links. *Geom. Topol.*, 10:695–713, 2006.
- [30] P. Ozsváth and Z. Szabó. Heegaard Floer homologies and contact structures. 129(1):39–61, 2005.
- [31] P. Ozsváth and Z. Szabó. Holomorphic disks, link invariants, and the multi-variable Alexander polynomial. 8:615–692, 2008.
- [32] P. Ozsváth, Z. Szabó, and D. Thurston. Legendrian knots, transverse knots, and combinatorial Floer homology. 12:941–980, 2008.
- [33] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334 (electronic), 2004.
- [34] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004.
- [35] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and link invariants. math.GT/0512286, 2005.
- [36] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.
- [37] Jacob Andrew Rasmussen. *Floer homology and knot complements*. PhD thesis, Harvard University, 2003.
- [38] Lawrence P. Roberts. On knot Floer homology in double branched covers. math.GT/0706.0741, 2007.

BOSTON COLLEGE; DEPARTMENT OF MATHEMATICS; 301 CARNEY HALL; CHESTNUT HILL, MA 02467

*E-mail address:* john.baldwin@bc.edu

BOSTON COLLEGE; DEPARTMENT OF MATHEMATICS; 301 CARNEY HALL; CHESTNUT HILL, MA 02467

*E-mail address:* julia.grigsby@bc.edu